Journal of Engineering Mathematics **32:** 41–52, 1997.

Elastic fields produced by a point source in solids of general anisotropy

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Received 21 December 1995; accepted in revised form 22 November 1996

Abstract. Explicit expressions for three-dimensional elastostatic Green's functions in solids of general anisotropy are derived by means of an integral-representation technique and a subsequent application of the residue calculus. A direct calculation for the derivatives of the displacement Green's functions is quite complicated. However, relatively simple expressions can be obtained for the integration of the derivatives along a line or a surface. These integrals are in fact more useful, because, in most applications, not the derivatives, but their integrals along lines or surfaces are needed. Discussions regarding degenerate materials and details in evaluation of the residues at multiple poles are also given. The results presented in this paper are sufficient for the implementation of the boundary-element method for bodies of general anisotropic solids.

Key words: anisotropic solid, Green's function, elastostatics, boundary-element method, integral equation

1. Introduction

The elastic response of an unbounded homogeneous linearly elastic solid to a time-independent point load provides a basic building block for the formulation of integral representations and boundary-integral equations, and for the solution of the latter by the boundary-element method. This three-dimensional basic solution, which is often called the elastostatic Green's function, is also essential in the construction of elastodynamics solutions (Wang and Achenbach [1, 2]). For *isotropic solids* both the two-dimensional (2-D) and three-dimensional (3-D) Green's functions are classic and can be found in many text books. For general *anisotropic solids*, the 2-D Green's functions have been extensively studied (Barnett and Lothe [3, 4], Hwu and Yen [5], Stroh [6, 7], Ting [8, 9] and Wang [10]). The 3-D Green's functions in the form of contour integrals have been investigated by Barnett and Swanger [11], Condat and Kirchner [12], Fredholm [13], Gyndersen and Lothe [14], Lifshitz and Rozenzweig [15], Synge [16] and Mura [17]. These integral expressions have been used to solve boundary-value problems by the boundary-element method by Vogel and Rizzo [18], Wilson and Cruse [19] and DiNicola [20]. Much effort has also been devoted to deriving explicit expressions. For the special case of transversely isotropic materials, explicit expressions have been obtained by Lifshitz and Rozenzweig [15], Elliott [21], Kroner [22], Willis [23], Lejcek [24] and Pan and Chou [25]. For solids of general anisotropy, however, explicit expressions for the 3-D Green's functions are still not available in the literature.

To derive these expressions for general anisotropic solids is the objective of this paper (part of the material presented here has been published in the form of conference proceedings [26, 27]). The first issue of this paper is to obtain explicit expressions for the displacement Green's functions by means of the plane integral representation for the delta function. The solutions are expressed in terms of residues of poles whose positions are given by the roots of the sextic equation of elasticity. The second issue is the calculation of the derivatives of the displacement Green's functions. We note that knowing only the displacement field is not enough. In most applications, both the displacement and the stress Green's functions are involved. In calculations of cracks and dislocations, we also need higher-order derivatives of the displacement Green's functions. These derivatives are, however, very difficult to obtain. In deriving solutions for stress fields produced by a dislocation loop in anisotropic solids, Willis [28] showed that, even though the derivatives of the displacement Green's functions are complicated, their integrals along a line segment may have relatively simple solutions. Using a different approach, Wang [29] reconfirmed Willis's observation and corrected mistakes in Willis's derivation. The results of Willis [28] and Wang [29] are of general interest because in most computations solutions of integration of the Green's functions over curves or surfaces are needed. Employing the technique of [29], we obtain relatively simple expressions for line integrals of the first-order derivatives of the displacement Green's functions. Following the same approach, we also calculate surface integrals of the second-order derivatives. Discussions regarding degenerate materials and details in evaluation of the residues at multiple poles are also given. The results presented in this paper should be sufficient for the implementation of the boundary-element method for solids of general anisotropy.

2. Definition of the displacement Green's function

Consider an unbounded homogeneous anisotropic linearly elastic solid subjected to a point load in a fixed rectangular coordinate system. Denoted by $g_{pk}(\boldsymbol{x})$, the displacement Green's function corresponds to the displacement field at point $\boldsymbol{x}$ in the x_p-direction produced by a point load applied at the origin in the x_k-direction. Mathematically, $g_{pk}(\boldsymbol{x})$ is defined by the partial differential equations

$$\Gamma_{ip}(\partial_{\boldsymbol{x}})g_{pk}(\boldsymbol{x}) = -\delta_{ik}\delta(\boldsymbol{x}), \tag{1}$$

where δ_{ik} is the Kronecker delta, $\delta(\boldsymbol{x})$ the delta function, and

$$\Gamma_{ip}(\partial_{\boldsymbol{x}}) = c_{ijpq}\partial_j\partial_q. \tag{2}$$

The elastic constants c_{ijpq} are fully symmetric and positive definite, *i.e.*,

$$c_{ijpq} = c_{jipq} = c_{ijqp} = c_{pqij} \tag{3}$$

and

$$c_{ijpq}e_{ij}e_{pq} > 0 \tag{4}$$

for any non-zero tensor e_{ij}.

In this paper, both suffix notations and bold-face letters are used to describe three-dimensional vectors. The summation convention is assumed. Thus, $\boldsymbol{x}$ and $\boldsymbol{n}$ have components x_i and n_i ($i = 1, 2$, and 3), and $\boldsymbol{n}\cdot\boldsymbol{x} = n_i x_i$ is the vector inner product. For a function $f(\boldsymbol{x})$, the derivative with respect to x_i is written as $\partial_i f(\boldsymbol{x})$ or $f_{,i}(\boldsymbol{x})$.

3. Illustration of the solution method

We shall explain the solution method by carrying out an example for Laplace's equation:

$$\Delta g(\boldsymbol{x}) = -\delta(\boldsymbol{x}). \tag{5}$$

The starting point of the derivation is the use of the plane integral representation for $\delta(\boldsymbol{x})$:

$$\frac{-1}{8\pi^2}\Delta \int_{\Omega} \frac{1}{|\boldsymbol{n}|^2}\delta(\boldsymbol{n}\cdot\boldsymbol{x})\,\mathrm{d}\Omega(\boldsymbol{n}) = \delta(\boldsymbol{x}), \tag{6}$$

where Ω is any closed surface in a three-dimensional $\boldsymbol{n}$-space which encloses the origin point $\boldsymbol{n} = 0$. The proof of (6) and details about the plane integral representation for arbitrary functions can be found in texts on the Radon transform (John [30] and Gel'fand *et al.* [31]).

It follows from (6) that (5) is satisfied by

$$g(\boldsymbol{x}) = \frac{1}{8\pi^2}\int_{\Omega} \frac{1}{|\boldsymbol{n}|^2}\delta(\boldsymbol{n}\cdot\boldsymbol{x})\,\mathrm{d}\Omega(\boldsymbol{n}). \tag{7}$$

To evaluate the integral we expand the $\boldsymbol{n}$-space in the orthogonal bases $\boldsymbol{p}, \boldsymbol{q}$, and $\boldsymbol{e}$ as

$$\boldsymbol{n} = \boldsymbol{p}\xi + \boldsymbol{q}\zeta + \boldsymbol{e}\eta \tag{8}$$

with

$$\boldsymbol{e} = \boldsymbol{x}/r \quad (r = |\boldsymbol{x}|), \qquad \boldsymbol{p} = \frac{\boldsymbol{e}\times\boldsymbol{c}}{|\boldsymbol{e}\times\boldsymbol{c}|}, \quad \text{and} \quad \boldsymbol{q} = \boldsymbol{e}\times\boldsymbol{p}. \tag{9}$$

Here $\times$ denotes the vector (cross) product and $\boldsymbol{c}$ can be any vector but $|\boldsymbol{e}\times\boldsymbol{c}| \neq 0$. In doing so,

$$\boldsymbol{n}\cdot\boldsymbol{x} = \boldsymbol{p}\cdot\boldsymbol{x}\xi + \boldsymbol{q}\cdot\boldsymbol{x}\zeta + \boldsymbol{e}\cdot\boldsymbol{x}\eta = r\eta \tag{10}$$

and (7) changes to

$$g(\boldsymbol{x}) = \frac{1}{8\pi^2 r}\int_{\Omega} \frac{1}{|\boldsymbol{p}\xi + \boldsymbol{q}\zeta + \boldsymbol{e}\eta|^2}\delta(\eta)\,\mathrm{d}\Omega(\xi,\zeta,\eta). \tag{11}$$

where Ω is now any closed surface enclosing $\xi = \zeta = \eta = 0$. Let us choose Ω as the rectangular parallelepiped shown in Figure 1, and let us consider the case in which its size $L \to \infty$. The surfaces S_1 and S_2 are then defined by $\xi = \pm 1$. Over surfaces other than S_1 and S_2, the integrand of (11) approaches zero as $1/L^2$, and the corresponding integrations yield zero values. The integral surface Ω thus may be replaced by the planes S_1 and S_2. Furthermore, it follows by the symmetry of the integrand that the integration over S_1 equals that over S_2. Therefore, (11) may be reduced to an integral over S_1 $(\xi = 1)$:

$$g(\boldsymbol{x}) = \frac{1}{4\pi^2 r}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{1}{|\boldsymbol{p} + \boldsymbol{q}\zeta + \boldsymbol{e}\eta|^2}\delta(\eta)\,\mathrm{d}\zeta\,\mathrm{d}\eta. \tag{12}$$

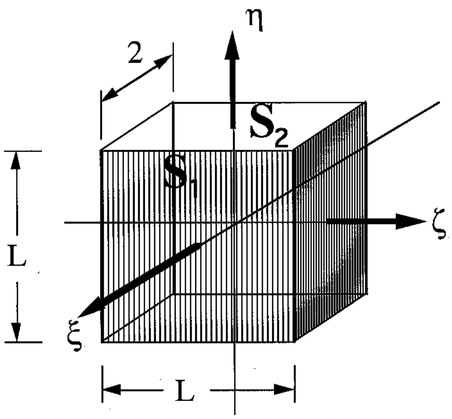

Figure 1. Integral contour over a rectangular parallelepiped.

Integration with respect to η yields

$$g(\boldsymbol{x}) = \frac{1}{4\pi^2 r}\int_{-\infty}^{\infty}\frac{1}{|\boldsymbol{p}+\boldsymbol{q}\zeta|^2}\,\mathrm{d}\zeta = \frac{1}{4\pi^2 r}\int_{-\infty}^{\infty}\frac{1}{1+\zeta^2}\,\mathrm{d}\zeta. \tag{13}$$

The integral can be expressed in terms of the residue of the pole at $\zeta = i$. The result is

$$g(\boldsymbol{x}) = \frac{1}{4\pi}\frac{1}{r}. \tag{14}$$

Equation (14) is the well-known solution for the Green's function of Laplace's equation.

4. Explicit expressions for the displacement Green's functions

We now return to the Green's function defined by (1). Since

$$\partial_p f(\boldsymbol{n}\cdot\boldsymbol{x}) = n_p \dot{f}(\boldsymbol{n}\cdot\boldsymbol{x}), \tag{15}$$

where a dot denotes differentiation with respect to the argument, we have

$$\Gamma_{ip}(\partial_{\boldsymbol{x}})\int_\Omega \Gamma_{pk}^{-1}(\boldsymbol{n})\delta(\boldsymbol{n}\cdot\boldsymbol{x})\,\mathrm{d}\Omega(\boldsymbol{n}) = \delta_{ik}\Delta\int_\Omega \frac{1}{|\boldsymbol{n}|^2}\delta(\boldsymbol{n}\cdot\boldsymbol{x})\,\mathrm{d}\Omega(\boldsymbol{n}), \tag{16}$$

where $\Gamma_{pk}^{-1}(\boldsymbol{n})$ is the inverse matrix of

$$\Gamma_{ip}(\boldsymbol{n}) = c_{ijpq}n_j n_q. \tag{17}$$

By virtue of (3) and (4) $\Gamma_{ip}(\boldsymbol{n})$ is symmetric and positive definite, so that $\Gamma_{pk}^{-1}(\boldsymbol{n})$ is well-defined. It follows from (6) and (16) that (1) is satisfied by

$$g_{pk}(\boldsymbol{x}) = \frac{1}{8\pi^2}\int_\Omega \Gamma_{pk}^{-1}(\boldsymbol{n})\delta(\boldsymbol{n}\cdot\boldsymbol{x})\,\mathrm{d}\Omega(\boldsymbol{n}) \tag{18}$$

$$= \frac{1}{8\pi^2}\int_\Omega \frac{A_{pk}(\boldsymbol{n})}{D(\boldsymbol{n})}\delta(\boldsymbol{n}\cdot\boldsymbol{x})\,\mathrm{d}\Omega(\boldsymbol{n}), \tag{19}$$

where

$$A_{pk}(\boldsymbol{n}) = \text{adj}[\Gamma_{pk}(\boldsymbol{n})], \qquad D(\boldsymbol{n}) = \det[\Gamma_{pk}(\boldsymbol{n})]. \tag{20}$$

Integral expressions similar to (19) can be found in [11]–[17] and [21, 22]. Now, following the steps shown in going from (7) to (13), we obtain

$$g_{pk}(\boldsymbol{x}) = \frac{1}{4\pi^2 r} \int_{-\infty}^{\infty} \frac{A_{pk}(\boldsymbol{p} + \zeta\boldsymbol{q})}{D(\boldsymbol{p} + \zeta\boldsymbol{q})}\, \mathrm{d}\zeta. \tag{21}$$

It follows from (20) that $A_{pk}(\boldsymbol{p}+\zeta\boldsymbol{q})$ and $D(\boldsymbol{p}+\zeta\boldsymbol{q})$ are polynomials of ζ of order four and six, respectively. To evaluate the integral (21) by residue calculus, we need to know the poles located at the roots of $D(\boldsymbol{p}+\zeta\boldsymbol{q})$. Since $\Gamma_{pk}^{-1}(\boldsymbol{n})$ is well-defined, $D(\boldsymbol{p}+\zeta\boldsymbol{q})$ does not have real roots. We also know that a polynomial of order N with real coefficients has N roots, and if $a+ib$ is a root $a-ib$ must also be a root. Consequently, there are three roots ζ_m satisfying

$$D(\boldsymbol{p} + \zeta_m\boldsymbol{q}) = 0 \tag{22}$$

with

$$\text{Im}\,\zeta_m > 0 \quad (m = 1, 2, 3) \tag{23}$$

and we may write

$$D(\boldsymbol{p} + \zeta\boldsymbol{q}) = \sum_{k=0}^{6} a_k\zeta^k = a_6 \prod_{m=1}^{3} (\zeta - \zeta_m)(\zeta - \zeta_m^*), \tag{24}$$

where ζ_m^* are the conjugates of ζ_m and a_k are the coefficients of the polynomial $D(\boldsymbol{p}+\zeta\boldsymbol{q})$. Equation (22) is called the sextic equation of elasticity (Head [32]). It is useful to note that $a_6 = D(\boldsymbol{q})$.

Now, expressing the integral (21) in terms of the residues of poles located at ζ_m, we get

$$g_{pk}(\boldsymbol{x}) = \frac{1}{4\pi^2 r} \sum_{m=1}^{3} \left[2\pi i \frac{A_{pk}(\boldsymbol{p} + \zeta\boldsymbol{q})}{\partial_\zeta D(\boldsymbol{p} + \zeta\boldsymbol{q})} \right]_{\text{at } \zeta=\zeta_m}. \tag{25}$$

where $\partial_\zeta D(\boldsymbol{p}+\zeta\boldsymbol{q})$ can be easily calculated from (24). Since integral (21) is real-valued, its solution (25) must also be real-valued. Thus, the summation of the imaginary parts of all the residues must be zero. Hence,

$$g_{pk}(\boldsymbol{x}) = \frac{-\text{Im}}{2\pi r} \sum_{m=1}^{3} \overline{g}_{pk}^{m}(\boldsymbol{e}), \tag{26}$$

where

$$\overline{g}_{pk}^{m}(\boldsymbol{e}) = \left[\frac{A_{pk}(\boldsymbol{p} + \zeta\boldsymbol{q})}{\partial_\zeta D(\boldsymbol{p} + \zeta\boldsymbol{q})} \right]_{\text{at } \zeta=\zeta_m} = \left[\frac{A_{pk}(\boldsymbol{n})}{\boldsymbol{q} \cdot \partial_{\boldsymbol{n}} D(\boldsymbol{n})} \right]_{\text{at } \boldsymbol{n}=\boldsymbol{p}+\zeta_m\boldsymbol{q}}. \tag{27}$$

As usual, for a complex value $z = a + ib, \mathrm{Re}(z) = a$ and $\mathrm{Im}(z) = b$. In (26), $-\mathrm{Im}$ comes from the fact that $\mathrm{Re}(iz) = -\mathrm{Im}(z)$.

In deriving (26) we have assumed that ζ_m are distinct roots of (22). For certain materials, *e.g.*, isotropic solids, ζ_m are not all distinct. For such cases, we say that the solid is degenerate. For degenerate solids (27) must be modified, since the denominator $\partial_\zeta D(\boldsymbol{p} + \zeta \boldsymbol{q}) = 0$ at $\zeta = \zeta_m$. Modification for degenerated solids will be given in Section 7. In practical numerical computations, however, we can obtain satisfactory results by slightly changing the elastic constants c_{ijpq} such that all ζ_m become distinct [2].

5. Line integrals of $g_{pk,i}$

It follows from (19) that

$$g_{pk,i}(\boldsymbol{x}) = \frac{1}{8\pi^2} \int_\Omega \frac{n_i A_{pk}(\boldsymbol{n})}{D(\boldsymbol{n})} \dot{\delta}(\boldsymbol{n} \cdot \boldsymbol{x}) \, \mathrm{d}\Omega(\boldsymbol{n}). \tag{28}$$

Explicit solutions of $g_{pk,i}$ are extremely complicated. However, the convolution of $g_{pk,i}$ with an arbitrary function f along a curve $\boldsymbol{x}(l)$:

$$I_{pki}(\boldsymbol{y}) = \int_a^b g_{pk,i}[\boldsymbol{x}(l) - \boldsymbol{y}] f(l) \, \mathrm{d}l \tag{29}$$

can be calculated without the need for us to evaluate $g_{pk,i}$ directly. It follows from (29) that $\sigma_{ijk} = -c_{ijpq} I_{pkq}$ has the physical meaning of the stress field produced by a line force $f(l)$ acting in the x_k direction along the curve $\boldsymbol{x}(l)$.

The idea here is to put $g_{pk,i}$ into following form (Wang [29])

$$g_{pk,i}[\boldsymbol{x}(l) - \boldsymbol{y}] = \frac{\mathrm{d}}{\mathrm{d}l} V_{pki}(\boldsymbol{y}, l) + W_{pki}(\boldsymbol{y}, l). \tag{30}$$

We note that, referring to (40), the differentiation with respect to l will be cancelled upon substitution in (29), and thus it needs not be carried out. In (30),

$$V_{pki} = \frac{1}{8\pi^2} \int_\Omega \frac{n_i A_{pk}(\boldsymbol{n})}{(\boldsymbol{n} \cdot \boldsymbol{v}) D(\boldsymbol{n})} \delta(\boldsymbol{n} \cdot \overline{\boldsymbol{x}}) \, \mathrm{d}\Omega(\boldsymbol{n}), \tag{31}$$

and

$$W_{pki} = \frac{\kappa}{8\pi^2} \int_\Omega \frac{(\boldsymbol{n} \cdot \boldsymbol{w}) s_i A_{pk}(\boldsymbol{n})}{(\boldsymbol{n} \cdot \boldsymbol{v})^2 D(\boldsymbol{n})} \delta(\boldsymbol{n} \cdot \overline{\boldsymbol{x}}) \, \mathrm{d}\Omega(\boldsymbol{n}), \tag{32}$$

where

$$\overline{\boldsymbol{x}} \equiv \boldsymbol{x}(l) - y, \qquad \boldsymbol{v} \equiv \dot{\boldsymbol{x}}, \qquad \kappa \equiv |\ddot{\boldsymbol{x}}| = |\dot{\boldsymbol{v}}|, \quad \text{and} \quad \boldsymbol{w} \equiv \frac{1}{\kappa} |\ddot{\boldsymbol{x}}| = \frac{1}{\kappa} |\dot{\boldsymbol{v}}|. \tag{33}$$

The integrals in (31) and (32) are defined in the sense of Cauchy's principal value or the Hadamard finite part. Now, it is expedient to expand the $\boldsymbol{n}$-space by (6) but with $(\boldsymbol{p}, \boldsymbol{q}, \boldsymbol{e})$ being

$$\boldsymbol{e} = \overline{\boldsymbol{x}}/\overline{r} \quad (\overline{r} = |\overline{\boldsymbol{x}}|), \qquad \boldsymbol{p} = \frac{\boldsymbol{e} \times \boldsymbol{v}}{|\boldsymbol{e} \times \boldsymbol{v}|} \quad \text{and} \quad \boldsymbol{q} = \boldsymbol{e} \times \boldsymbol{p}. \tag{34}$$

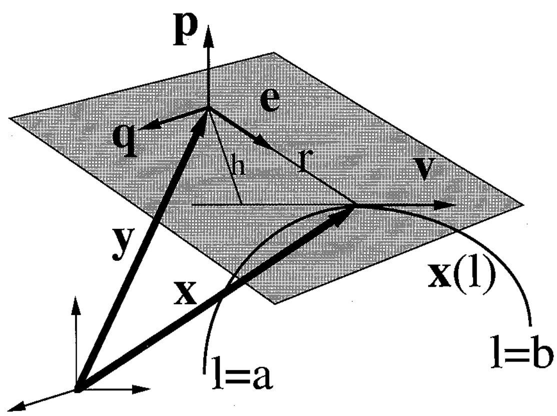

Figure 2. Geometry of $(\boldsymbol{p}, \boldsymbol{q}, \boldsymbol{e})$ for the integral of $g_{pk,i}$ along curve $\boldsymbol{x}(l)$.

The geometry of $(\boldsymbol{p}, \boldsymbol{q}, \boldsymbol{e})$ is illustrated in Figure 2. Again, following the steps shown in going from (7) to (13), we obtain

$$V_{pki} = \frac{1}{4\pi^2 h} \int_{-\infty}^{\infty} \frac{(p_i + q_i\zeta) A_{pk}(\boldsymbol{p} + \boldsymbol{q}\zeta)}{\zeta D(\boldsymbol{p} + \boldsymbol{q}\zeta)} \, \mathrm{d}\zeta, \tag{35}$$

and

$$W_{pki} = \frac{\kappa}{4\pi^2 h^2} \int_{-\infty}^{\infty} \frac{(\alpha + \zeta\beta)(p_i + q_i\zeta) A_{pk}(\boldsymbol{p} + \boldsymbol{q}\zeta)}{\zeta^2 D(\boldsymbol{p} + \boldsymbol{q}\zeta)} \, \mathrm{d}\zeta, \tag{36}$$

where

$$\alpha = (\boldsymbol{p} \cdot \boldsymbol{w})\overline{r}, \qquad \beta = (\boldsymbol{q} \cdot \boldsymbol{w})\overline{r} \quad \text{and} \quad h = (\boldsymbol{q} \cdot \boldsymbol{v})\overline{r}. \tag{37}$$

Geometrically, $h(l)$ corresponds to the perpendicular distance from $\boldsymbol{y}$ to the tangent line of $\boldsymbol{x}(l)$ (see Figure 2).

The pole corresponding to $\zeta = 0$ is real-valued. Hence its residue is purely imaginary and gives no contribution. Accordingly

$$V_{pki} = \frac{-\mathrm{Im}}{2\pi} \frac{1}{h} \sum_{m=1}^{3} [(p_i \zeta_m^{-1} + q_i) \overline{g}_{pk}^{m}] \tag{38}$$

and

$$W_{pki} = \frac{-\mathrm{Im}}{2\pi} \frac{\kappa}{h^2} \sum_{m=1}^{3} [(\alpha\zeta_m^{-1} + \beta)(p_i \zeta_m^{-1} + q_i) \overline{g}_{pk}^{m}]. \tag{39}$$

Now, substituting (30) in (29) and integrating by part, we obtain

$$\begin{aligned} I_{pki} &= V_{pki}(\boldsymbol{y}, b) f(b) - V_{pki}(\boldsymbol{y}, a) f(a) \\ &\quad + \int_a^b [W_{pki}(\boldsymbol{y}, l) f(l) - V_{pki}(\boldsymbol{y}, l) \dot{f}(l)] \, \mathrm{d}l. \end{aligned} \tag{40}$$

For a straight line, $\kappa = 0$ and thus $W_{pki} = 0$. Hence, for a constant f, Equation (40) reduces to the following explicit form

$$I_{pki} = [V_{pki}(\boldsymbol{y}, b) - V_{pki}(\boldsymbol{y}, a)]f. \tag{41}$$

6. Surface integrals of $g_{pk,ij}$

Analogous to the case of $g_{pk,i}$, we can obtain the convolution of $g_{pk,ij}$ with an arbitrary function f over a surface $\boldsymbol{x}(l_1, l_2)$:

$$J_{pkij}(\boldsymbol{y}) = \int_{a_2}^{b_2}\int_{a_1}^{b_1} g_{pk,ij}[\boldsymbol{x}(l_1, l_2) - \boldsymbol{y}]f(l_1, l_2)\, \mathrm{d}l_1\, \mathrm{d}l_2 \tag{42}$$

without directly carrying out complicated derivations for

$$g_{pk,ij}(\boldsymbol{x} - \boldsymbol{y}) = \frac{1}{8\pi^2}\int_{\Omega} \frac{n_i n_j A_{pk}(\boldsymbol{n})}{D(\boldsymbol{n})} \ddot{\delta}[\boldsymbol{n} \cdot (\boldsymbol{x} - \boldsymbol{y})]\, \mathrm{d}\Omega(\boldsymbol{n}). \tag{43}$$

Again, the idea is to put

$$g_{pk,ij}[\boldsymbol{x}(l_1, l_2) - \boldsymbol{y}] = \partial_{l_1}\partial_{l_2} S^0_{pkij} + \partial_{l_2} S^1_{pkij} + \partial_{l_1} S^2_{pkij} + S^3_{pkij} \tag{44}$$

and to obtain explicit solutions for S^N_{pkij} $(N = 0 \sim 3)$. It can be shown that

$$S^0_{pkij}(\boldsymbol{y}; l_1, l_2) = \frac{1}{8\pi^2}\int_{\Omega} \frac{n_i n_j A_{pk}(\boldsymbol{n})}{(\boldsymbol{v}_1 \cdot \boldsymbol{n})(\boldsymbol{v}_2 \cdot \boldsymbol{n}) D(\boldsymbol{n})} \delta(\boldsymbol{n} \cdot \overline{\boldsymbol{x}})\, \mathrm{d}\Omega(\boldsymbol{n}) \tag{45}$$

where

$$\boldsymbol{v}_1 = \partial_{l_1}\boldsymbol{x}(l_1, l_2) \quad \text{and} \quad \boldsymbol{v}_2 = \partial_{l_2}\boldsymbol{x}(l_1, l_2). \tag{46}$$

Following the routine steps shown in the previous sections, we obtain

$$S^0_{pkij} = \frac{-\mathrm{Im}}{2\pi r}\sum_{m=1}^{3} R^m_{ij}\overline{g}^m_{pk} \tag{47}$$

where

$$R^m_{ij} = \left[\frac{n_i n_j}{(\boldsymbol{v}_1 \cdot \boldsymbol{n})(\boldsymbol{v}_2 \cdot \boldsymbol{n})}\right]_{\text{at } \boldsymbol{n} = \boldsymbol{p} + \zeta_m \boldsymbol{q}}. \tag{48}$$

and where $\boldsymbol{p}$ and $\boldsymbol{q}$ are defined by (9).

Derivations for other terms in (44) are straightforward, but the results are quite lengthy for an arbitrary surface. When the surface is a rectangle, $S^1_{pkij} = S^2_{pkij} = S^3_{pkij} = 0$, and the integral (42) has the following explicit solution

$$\begin{aligned} J_{pkij}(\boldsymbol{y}) &= [S^0_{pkij}(\boldsymbol{y}; b_1, b_2) - S^0_{pkij}(\boldsymbol{y}; b_1, a_2) \\ &\quad - S^0_{pkij}(\boldsymbol{y}; a_1, b_2) + S^0_{pkij}(\boldsymbol{y}; a_1, a_2)]f \end{aligned} \tag{49}$$

for a constant f.

7. Degenerate materials

So far, in evaluating the residues, we have assumed that poles located at the roots of $\partial_\zeta D(\boldsymbol{p}+\zeta\boldsymbol{q})=0$ are all distinct, *i.e.*, $\zeta_1 \neq \zeta_2 \neq \zeta_3$. It is not true if the material is degenerate. Generally we must consider simple poles as well as multiple poles. Consider the most general case where there are $M(1 \leqslant M \leqslant 3)$ distinct poles at ζ_m of order N_m. Then, integral (21) yields the sum of the residues given by

$$g_{pk}(\boldsymbol{x}) = \frac{-\mathrm{Im}}{2\pi r}\sum_{m=1}^{M}\partial_\zeta^{N_m-1}\left[\frac{A_{pk}(\boldsymbol{p}+\zeta\boldsymbol{q})}{\widetilde{D}_m(\zeta)}\right]_{\text{at }\zeta=\zeta_m}, \tag{50}$$

where

$$\widetilde{D}_m(\zeta) = (N_m-1)!(\zeta-\zeta_m)^{-N_m}D(\boldsymbol{p}+\zeta\boldsymbol{q}). \tag{51}$$

Clearly, this is the solution for the most general anisotropy, including non-degenerate and degenerate materials. General solutions for V_{pki} and W_{pki} defined by integrals (35) and (36) as well as those for S^N_{pkij} discussed in Section 6 can be expressed in the same form.

Let us explore some details for degenerate materials. When $\zeta_1 = \zeta_2 \neq \zeta_3$, (50) corresponds to

$$g_{pk}(\boldsymbol{x}) = \frac{-\mathrm{Im}}{2\pi r}\left\{\partial_\zeta\left[\frac{A_{pk}(\boldsymbol{p}+\zeta\boldsymbol{q})}{\widetilde{D}_1(\zeta)}\right]_{\text{at }\zeta=\zeta_1} + \left[\frac{A_{pk}(\boldsymbol{p}+\zeta_3\boldsymbol{q})}{\widetilde{D}_3(\zeta_3)}\right]\right\}, \tag{52}$$

where, according to (51) and (24),

$$\begin{aligned}\widetilde{D}_1(\zeta) &= a_6(\zeta-\zeta_1^*)^2(\zeta-\zeta_2)(\zeta-\zeta_2^*),\\ \widetilde{D}_3(\zeta) &= a_6(\zeta-\zeta_1)^2(\zeta-\zeta_1^*)^2(\zeta-\zeta_3^*).\end{aligned} \tag{53}$$

When $\zeta_1 = \zeta_2 = \zeta_3$, we have

$$g_{pk}(\boldsymbol{x}) = \frac{-\mathrm{Im}}{2\pi r}\left\{\partial_\zeta^2\left[\frac{A_{pk}(\boldsymbol{p}+\zeta\boldsymbol{q})}{\widetilde{D}_1(\zeta)}\right]_{\text{at }\zeta=\zeta_1}\right\}, \tag{54}$$

where

$$\widetilde{D}_1(\zeta) = 2a_6(\zeta-\zeta_1^*)^3. \tag{55}$$

A direct calculation for the differentiation in (52) and (54) is mathematically straightforward, but by no means easy generally. This calculation can be simplified somewhat when we use the following integrals:

$$\alpha_{pk} = \int_C \frac{A_{pk}(\boldsymbol{p}+\zeta\boldsymbol{q})}{D(\boldsymbol{p}+\zeta\boldsymbol{q})}\,\mathrm{d}\zeta \quad \text{and} \quad \beta_{pk} = \int_C \frac{\zeta A_{pk}(\boldsymbol{p}+\zeta\boldsymbol{q})}{D(\boldsymbol{p}+\zeta\boldsymbol{q})}\,\mathrm{d}\zeta, \tag{56}$$

where C is a closed contour enclosing all the poles at ζ_m and ζ_m^* in the complex ζ-plane. We may express the solutions of these integrals in terms of the residues of poles inside the contour, or, equivalently, in terms of the residues of the poles at infinity. Thus, we have the following identities

$$\alpha_{pk} = \sum_{m=1}^{M} \left[\operatorname*{Res}_{\zeta=\zeta_m} \frac{A_{pk}(\boldsymbol{p}+\zeta\boldsymbol{q})}{D(\boldsymbol{p}+\zeta\boldsymbol{q})} + \operatorname*{Res}_{\zeta=\zeta_m^*} \frac{A_{pk}(\boldsymbol{p}+\zeta\boldsymbol{q})}{D(\boldsymbol{p}+\zeta\boldsymbol{q})} \right] = 0, \tag{57}$$

$$\begin{aligned} \beta_{pk} &= \sum_{m=1}^{M} \left[\operatorname*{Res}_{\zeta=\zeta_m} \frac{\zeta A_{pk}(\boldsymbol{p}+\zeta\boldsymbol{q})}{D(\boldsymbol{p}+\zeta\boldsymbol{q})} + \operatorname*{Res}_{\zeta=\zeta_m^*} \frac{\zeta A_{pk}(\boldsymbol{p}+\zeta\boldsymbol{q})}{D(\boldsymbol{p}+\zeta\boldsymbol{q})} \right] \\ &= 2\pi i \frac{A_{pk}(\boldsymbol{q})}{D(\boldsymbol{q})}. \end{aligned} \tag{58}$$

In the case $\zeta_1 = \zeta_2 \neq \zeta_3$, some simple direct manipulations on (57) and (58) leads to

$$\operatorname{Re}\left(\partial_\zeta \left[\frac{A_{pk}(\boldsymbol{p}+\zeta\boldsymbol{q})}{\widetilde{D}_1(\zeta)} \right]_{\text{at } \zeta=\zeta_1} \right) = -\operatorname{Re}(Q_{pk}), \tag{59}$$

$$\begin{aligned} &\operatorname{Im}\left(\partial_\zeta \left[\frac{A_{pk}(\boldsymbol{p}+\zeta\boldsymbol{q})}{\widetilde{D}_1(\zeta)} \right]_{\text{at } \zeta=\zeta_1} \right) \\ &\quad = \frac{1}{\operatorname{Im}\zeta_1} [-\operatorname{Re}(\zeta_1)\operatorname{Re}(Q_{pk}) + \operatorname{Re}(P_{pk} + \zeta_2 Q_{pk}) - 0{\cdot}5\gamma_{pk}], \end{aligned} \tag{60}$$

where

$$\begin{aligned} &P_{pk} = \left[\frac{A_{pk}(\boldsymbol{p}+\zeta_1\boldsymbol{q})}{\widetilde{D}_1(\zeta_1)} \right], \qquad Q_{pk} = \left[\frac{A_{pk}(\boldsymbol{p}+\zeta_3\boldsymbol{q})}{\widetilde{D}_3(\zeta_3)} \right], \\ &\gamma_{pk} = \left[\frac{A_{pk}(\boldsymbol{q})}{D(\boldsymbol{q})} \right]. \end{aligned} \tag{61}$$

Similarly, when $\zeta_1 = \zeta_2 = \zeta_3$, (57) and (58) yield

$$\operatorname{Re}\left(\partial_\zeta^2 \left[\frac{A_{pk}(\boldsymbol{p}+\zeta\boldsymbol{q})}{\widetilde{D}_1(\zeta)} \right]_{\text{at } \zeta=\zeta_1} \right) = 0, \tag{62}$$

$$\operatorname{Im}\left(\partial_\zeta^2 \left[\frac{A_{pk}(\boldsymbol{p}+\zeta\boldsymbol{q})}{\widetilde{D}_1(\zeta)} \right]_{\text{at } \zeta=\zeta_1} \right) = \frac{1}{\operatorname{Im}\zeta_1} [2\operatorname{Re}(R_{pk}) - 0{\cdot}5\gamma_{pk}], \tag{63}$$

where

$$R_{pk} = \partial_\zeta \left[\frac{A_{pk}(\boldsymbol{p}+\zeta\boldsymbol{q})}{\widetilde{D}_1(\zeta)} \right]_{\text{at } \zeta=\zeta_1}. \tag{64}$$

It is known that when $\zeta_1 = \zeta_2 = \zeta_3$,

$$A_{pk}(\boldsymbol{p} + \zeta_1 \boldsymbol{q}) = 0 \tag{65}$$

holds for at least all the solids known to us (see Ting [8]). It follows that

$$R_{pk} = \left[\frac{\partial_\zeta A_{pk}(\boldsymbol{p} + \zeta \boldsymbol{q})}{\widetilde{D}_1(\zeta)} \right]_{\text{at } \zeta=\zeta_1}. \tag{66}$$

We mention again that the modification for degenerate materials presented in this section is of interest in the sense of completeness of the analysis. They are not, however, necessarily essential to practical numerical computations. The most efficient way is to change the elastic constants c_{ijpq} slightly such that all ζ_m become distinct.

Summary and remarks

Based on the use of the plane integral representation of the delta function and a subsequent application of the residue calculus, elastostatic Green's functions for solids of general anisotropy were derived. The results presented in this paper should be sufficient for the implementation of the boundary-element method. A simple example related to the Laplace equation was worked out in Section 3 illustrating the solution method. The method was then used to obtain explicit expressions for the displacement Green's functions in Section 4. The first-order derivatives of the displacement Green's functions were discussed in Section 5. It was found that, although a direct evaluation of the derivatives is complicated, relatively simple expressions can be obtained for the integration of the derivatives along a line. These integrals are in fact more useful, because in most applications computations are carried out, not for the derivatives, but for their integrals along a line or surface. Following the same approach, we derived surface integrals of the second-order derivatives in Section 6. The calculations for the second-order derivatives were, however, not necessary in the implementation of the boundary-element method, because it was found by the author that the boundary-integral equations involving the second-order derivatives can always be transformed into those with only the displacement Green's functions and their first-order derivatives. Results in this regard will be reported in a forthcoming paper. In the last section, discussions regarding degenerate materials and details in evaluation of the residues at multiple poles were also given. It was emphasized, however, that the modification for degenerate materials is not necessarily essential to practical numerical computations. The most efficient way is to change the elastic constants slightly such that the material becomes non-degenerate.

Acknowledgments

Helpful comments by Professor G.S. Miller are gratefully acknowledged. Thanks also go to Professor Barnett and Professor Mura for providing part of the references.

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